

ON THE MOMENT MAP ON SYMPLECTIC MANIFOLDS

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ABSTRACT. We consider a connected symplectic manifold M acted on by a connected Lie group G in a Hamiltonian fashion. If G is compact, we prove give an Equivalence Theorem for the symplectic manifolds whose squared moment map $\|\mu\|^2$ is constant. This result works also in the almost-Kähler setting. Then we study the case when G is a non compact Lie group acting properly on M and we prove a splitting results for symplectic manifolds.

1. INTRODUCTION

We shall consider symplectic manifolds (M, ω) acted on by a connected Lie group G of symplectomorphism. Throughout this paper we shall assume that the G -action is proper and Hamiltonian, i.e. there exists a moment map $\mu : M \longrightarrow \mathfrak{g}^*$, where \mathfrak{g} is the Lie algebra of G . In general the matter of existence/uniqueness of μ is delicate. However, if \mathfrak{g} is semisimple, there is a unique moment map (see [7]). If (M, ω) is a compact Kähler manifold and G is a connected compact Lie group of holomorphic isometries, then the existence problem is resolved (see [9]): a moment map exists if and only if G acts trivially on the Albanese torus $\text{Alb}(M)$ or equivalently every vector field from \mathfrak{z} , where \mathfrak{z} is the Lie algebra of the center of G , vanishes at some point in M .

If G is compact, we fix an $\text{Ad}(G)$ -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} and we identify \mathfrak{g}^* with \mathfrak{g} by means of $\langle \cdot, \cdot \rangle$. Then we can think of μ as a \mathfrak{g} -valued map and it is natural study the smooth function $f = \|\mu\|^2$ which has been extensively used in [10] to obtain strong information on the topology of the manifold.

Firstly, we investigate the symplectic manifolds whose squared moment map is constant proving the the following Equivalence Theorem.

Theorem 1.1. (*Equivalence Theorem*). *Suppose (M, ω) is a connected symplectic G -Hamiltonian manifold, where G is a connected compact Lie group acting effectively on M , with moment map $\mu : M \longrightarrow \mathfrak{g}$. Then the following conditions are equivalent.*

- (1) *G is semisimple and M is G -equivariantly symplectomorphic to a product of a flag manifold and a symplectic manifold which is acted on trivially by G ;*
- (2) *the squared moment map $f = \|\mu\|^2$ is constant;*
- (3) *M is mapped by the moment map μ to a single coadjoint orbit;*
- (4) *all principal G -orbits are symplectic;*
- (5) *all G -orbits are symplectic.*

Moreover, given a G -invariant ω -compatible almost complex structure on M , the symplectomorphism in (1) turns out to be an isometry with respect to the induced Riemannian metric while (4) and (5) become: all G -orbits (resp. principal G -orbits) are complex.

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In order to prove the above theorem, we need the following result, which might have an independent interest.

Proposition 1.2. *Let (M, ω) be a symplectic manifold and let G be connected compact Lie group acting in a Hamiltonian fashion on M with moment map μ . Assume $x \in M$ realizes a local maximum of the smooth function $f = \|\mu\|^2$. Then $G \cdot x$ is symplectic and there exists a neighborhood Y_o of x such that $G \cdot (Y_o \cap \mu^{-1}(\mu(x)))$ is a symplectic submanifold which is G -equivariantly symplectomorphic to a flag manifold and a symplectic manifold which is acted on trivially by G . Moreover, if $x \in M$ realizes the maximum of $f = \|\mu\|^2$ or any $z \in \mu^{-1}(\mu(x))$ realizes a local maximum of $f = \|\mu\|^2$, then*

- (1) $\mu^{-1}(\mu(x))$ is a symplectic submanifold of M ;
- (2) $G \cdot \mu^{-1}(\mu(x))$ is a symplectic submanifold of M which is G -equivariantly symplectomorphic to $(Gx \times \mu^{-1}(\mu(x)), \omega|_{G \cdot x} + \omega|_{\mu^{-1}(\mu(x))})$.

These results generalize ones given in [6] and [2].

One may try to prove Proposition 1.2 assuming only $\text{Ad}(G)$ is compact; this means G is covered for a compact Lie group and a vector group which lies in the center (see [5]). However, if G acts properly on M , then the existence of a G -symplectic orbit implies that G must be compact. Indeed if $G \cdot x = G/G_x$ is symplectic, from Proposition 1, then $(G_x)^o$, the connected component of stabilizer group of x which contains the identity, coincides with $(G_{\mu(x)})^o$. Since $(Z(G))^o \subset (G_{\mu(x)})^o$, we conclude that G is compact.

Then we study the case when G is a non compact Lie group acting properly and in a Hamiltonian fashion on M and we prove the following result.

Theorem 1.3. *Let (M, ω) be a symplectic manifold and let G be a connected non compact Lie group acting properly and in a Hamiltonian fashion on M with moment map μ . Assume also $G \cdot \alpha$ is a locally closed coadjoint orbit for every $\alpha \in \mathfrak{g}^*$. Then the following conditions are equivalent.*

- (1) All G -orbits are symplectic;
- (2) all principal G -orbits are symplectic;
- (3) M is mapped by the moment map μ to a single coadjoint orbit;
- (4) let x be a regular point of M . Then $G \cdot x$ is a symplectic orbit, $\mu^{-1}(\mu(x))$ is a symplectic submanifold on which G_x acts trivially and the following G -equivariant application

$$\phi : G \cdot x \times \mu^{-1}(\mu(x)) \longrightarrow M, \quad \phi([gx, z]) = gz,$$

is surjective and satisfies

$$\phi^*(\omega) = \omega|_{G \cdot x} + \omega|_{\mu^{-1}(\mu(x))}.$$

If G is a reductive Lie group acting effectively on M , then G has to be semisimple and ϕ becomes a G -equivariant symplectomorphism. Moreover, if $N(G_x)/G_x$ is finite, given a G -invariant ω -compatible almost complex structure on M , the symplectomorphism in (4) turns out to be an isometry with respect to the induced Riemannian metric while (1) and (2) become: all G -orbits (resp. principal G -orbits) are complex.

The assumption $G \cdot \alpha$ is a locally closed coadjoint orbit is needed to applying the symplectic slice and the symplectic stratification of the reduced spaces given in [1]. Observe that the condition of a coadjoint orbit being locally closed is automatic for reductive group and for

their product with vector spaces. There exists an example of a solvable group due to Mautner [17] p.512, with non-locally closed coadjoint orbits.

Finally, as an immediate corollary of Theorem 1.1 and Theorem 1.3, we give the following splitting result.

Let G be a non compact semisimple Lie group. The Killing form B on \mathfrak{g} is a non-degenerate $\text{Ad}(G)$ -invariant bilinear form. Therefore, we may identify \mathfrak{g} with \mathfrak{g}^* by means of $-B$ and we may think μ as a \mathfrak{g} -valued map.

Corollary 1.4. *Let M be a symplectic manifold acted on by connected non compact semisimple Lie group G , properly and in a Hamiltonian fashion with moment map μ . If $f = \|\mu\|^2$ is constant and any element which lies in the image of the moment map μ is elliptic, then all G -orbits are symplectic and M is G -equivariantly symplectomorphic to a product of a flag manifold and a symplectic manifold which is acted on trivially by G . Moreover, if $N(G_x)/G_x$ is finite, given a G -invariant ω -compatible almost complex structure on M , the application ϕ turn out to be an isometry with respect to the induced Riemannian metric and all G -orbits are complex.*

2. PROOF OF THE MAIN RESULTS

Let M be a connected differential manifold equipped with a non-degenerate closed 2-form ω . The pair (M, ω) is called *symplectic manifold*. Here we consider a finite-dimensional connected Lie group acting smoothly and properly on M so that $g^*\omega = \omega$ for all $g \in G$, i.e. G acts as a group of canonical or symplectic diffeomorphism.

The G -action is called *Hamiltonian*, and we said that G acts in a Hamiltonian fashion on M or M is G -Hamiltonian, if there exists a map

$$\mu : M \longrightarrow \mathfrak{g}^*,$$

which is called moment map, satisfying:

- (1) for each $X \in \mathfrak{g}$ let
 - $\mu^X : M \longrightarrow \mathbb{R}$, $\mu^X(p) = \mu(p)(X)$, the component of μ along X ,
 - $X^\#$ be the vector field on M generated by the one parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\} \subseteq G$.

Then

$$d\mu^X = i_{X^\#}\omega,$$

i.e. μ^X is a Hamiltonian function for the vector field $X^\#$.

- (2) μ is G -equivariant, i.e. $\mu(gp) = \text{Ad}^*(g)(\mu(p))$, where Ad^* is the coadjoint representation on \mathfrak{g}^* .

Let $x \in M$ and $d\mu_x : T_x M \longrightarrow T_{\mu(x)}\mathfrak{g}^*$ be the differential of μ at x . Then

$$\text{Ker} d\mu_x = (T_x G \cdot x)^{\perp \omega} := \{v \in T_x M : \omega(v, w) = 0, \forall w \in T_x G \cdot x\}.$$

If we restrict μ to a G -orbit $G \cdot x$, then we have the following homogeneous fibration

$$\mu : G \cdot x \longrightarrow \text{Ad}^*(G) \cdot \mu(x)$$

and the restriction of the ambient symplectic form ω to the orbit $G \cdot x$ equals the pullback by the moment map μ of the symplectic form on the coadjoint orbit through $\mu(x)$:

$$(1) \quad \omega|_{G \cdot x} = \mu^*(\omega_{\text{Ad}^*(G) \cdot \mu(x)})|_{G \cdot x},$$

see [1] p. 211, where $\omega_{G \cdot \mu(x)}$ is the Kirillov-Konstant-Souriau (KKS) symplectic form on the coadjoint orbit of $\mu(x)$ in \mathfrak{g}^* . This implies the following well-known fact, see [7].

Proposition 2.1. *The orbit of G through $x \in M$ is symplectic if and only if the stabilizer group of x is an open subgroup of the stabilizer of $\mu(x)$ if and only if the moment map restricted to $G \cdot x$ into $G \cdot \mu(x)$, $\mu|_{G \cdot x} : G \cdot x \longrightarrow G \cdot \mu(x)$, is a covering map. In particular if G is compact or semisimple, then $G_x = G_{\mu(x)}$ so this implies that $\mu|_{G \cdot x} : G \cdot x \longrightarrow G \cdot \mu(x)$ is a diffeomorphism.*

Proof. The first affirmation follows immediately from (1). If G is compact or semisimple, then $G_{\mu(x)}$ is connected so this implies that the two stabilizer groups are the same. \square

We now give the proof of Proposition 1.2

Proof of Proposition 1.2. Let $\beta = \mu(x)$ and let G_x be the isotropy group at x . From the local normal for the moment map, see [1], [7], [14] and [16], there exists a neighborhood of the orbit $G \cdot x$ which is equivariantly symplectomorphic to a neighborhood Y_o of the zero section of $(Y = G \times_{G_x} (\mathfrak{q} \oplus V), \tau)$ with the G -moment map μ given by

$$\mu([g, m, v]) = Ad(g)(\beta + m + \mu_V(v)),$$

where \mathfrak{q} is a summand in the G_x -equivariant splitting $\mathfrak{g} = \mathfrak{g}_\beta \oplus \mathfrak{s} = \mathfrak{g}_x \oplus \mathfrak{q} \oplus \mathfrak{s}$ and μ_V is the moment map of the G_x -action on the symplectic subspace V of $((T_x G \cdot x)^{\perp_\omega}, \omega(x))$. Note that V is isomorphic to the quotient $((T_x G \cdot x)^{\perp_\omega} / ((T_x G \cdot x)^{\perp_\omega} \cap T_x G \cdot x))$.

In the sequel we denote by $\omega_V = \omega(x)|_V$ and shrinking Y_o if necessary, we may suppose that $[e, 0, 0]$ is a maximum of the smooth function $f = \|\mu\|^2$ in Y_o .

We first want to prove that $G \cdot x$ is symplectic. Then we shall prove $\mathfrak{q} = \{0\}$.

Let $m \in \mathfrak{q} - \{0\}$. Then for every $\lambda \in \mathbb{R}$ we have

$$f(e, \lambda m, 0) = \|\beta\|^2 + \lambda^2 \|m\|^2 + \lambda \langle m, \beta \rangle \leq \|\beta\|^2$$

and therefore

$$\lambda^2 \|m\|^2 + \lambda \langle m, \beta \rangle \leq 0,$$

for every $\lambda \in \mathbb{R}$ which is a contradiction. Hence $G \cdot x$ is symplectic and by Proposition 2.1 $G_x = G_\beta$.

Note that any $y \in Y_o^\beta = Y_o \cap \mu^{-1}(\beta)$ is a local maximum for f . Then $G_y = G_x$ for every $y \in Y_o^\beta$, i.e. $G \cdot y$ is symplectic, and a G -orbit through an element of Y_o^β intersects $\mu^{-1}(\beta)$ in at most one point. Indeed, if both $x \in Y_o^\beta$ and kx lie in $\mu^{-1}(\beta)$, then, by the G -equivariance of μ , we have $\mu(kx) = \beta = k\mu(x) = k\beta$, proving $k \in G_x$. From this it follows that the map

$$\phi : G \cdot x \times Y_o^\beta \longrightarrow G \cdot Y_o^\beta$$

is well-defined and bijective.

Let us now return to the local normal form. Let $y \in Y_o^\beta$. We know that $G_y = G_x$ and $G \cdot y$ is symplectic. Hence there exists a neighborhood of $G \cdot y$ which is G -invariant symplectomorphic to a neighborhood Y' of the zero section of $(Y = G \times_{G_x} V, \tau)$ with the G -moment map given by

$$\mu([g, v]) = Ad(g)(\beta + \mu_V(v)).$$

Shrinking Y' , if necessary, we may assume $Y' \subseteq Y_o$ and, see Proposition 13 in [1] p.216, the intersection of the set $\mu^{-1}(G \cdot \beta)$ with Y' is of the form

$$G \cdot \mu^{-1}(\beta) \cap Y' = \{[g, v] \in Y_o : \mu_V(v) = 0\}.$$

Let $Y^{(G_x)} = \{m \in Y : (G_m) = (G_x)\}$, i.e. G_m is G -conjugate to G_x . It is easy to check that

$$(2) \quad Y^{(G_x)} = G \times_{G_x} V^{G_x} \cong G/G_x \times V^{G_x},$$

where $V^{G_x} = \{x \in V : G_m = G_x\}$, and $\mu(Y^{(G_x)}) = G \cdot \beta$. Therefore, since $Y_o^\beta \subseteq M^{G_x}$, we have

$$(3) \quad Y' \cap \mu^{-1}(G \cdot \beta) = Y^{(G_x)} \cap Y' = (Y')^{(G_x)}$$

and

$$Y' \cap \mu^{-1}(\beta) = Y' \cap V^{G_x}.$$

This implies both Y_o^β and $G \cdot Y_o^\beta$ are symplectic submanifolds of M . Indeed, from the above discussion we conclude that $T_y Y_o^\beta = V^{G_x}$ and the tangent space at y of $G \cdot Y_o^\beta$ splits as

$$T_y G \cdot Y_o^\beta = T_y G \cdot y \oplus^{\perp_\omega} T_y Y_o^\beta.$$

Here we have used that $T_y Y_o^\beta \subset (T_y G \cdot y)^{\perp_\omega} = \text{Ker } d\mu_y$ and $G \cdot y$ is symplectic.

Since

$$(4) \quad \tau|_{G/G_x \times_{G_x} V^{G_x}} = \omega|_{G \cdot x} + (\omega_V)|_{V^{G_x}},$$

see Corollary 14 p. 217 [1], from (1), (2), (3), and (4) we obtain that ϕ is a symplectomorphism.

Now assume that $x \in M$ realizes the maximum of f or any $z \in \mu^{-1}(\mu(x))$ is a local maximum of f . Let $\beta = \mu(x)$. Using the same argument as before, we may prove $G \cdot z$ is symplectic, $G_z = G_x = G_\beta$ for every $z \in \mu^{-1}(\beta)$ and a G -orbit intersects $\mu^{-1}(\beta)$ in at most one point. It follows that the following application

$$\phi : G \cdot x \times \mu^{-1}(\beta) \longrightarrow G \cdot \mu^{-1}(\beta), \quad \phi(gG_x, z) = gz$$

is a G -equivariant diffeomorphism. We shall prove that ϕ is a symplectomorphism.

The set $\mu^{-1}(G\beta) \cap M^{(G_x)}$ is a manifold of constant rank and the quotient

$$(M_\beta)^{(G_x)} := (\mu^{-1}(G \cdot \beta) \cap M^{(G_x)})/G,$$

is a symplectic manifold, see Corollary 14 in [1]. Since $\mu^{-1}(\beta) \subset M^{G_x}$, we have

$$G \cdot \mu^{-1}(\beta) = \mu^{-1}(G \cdot \beta) \cap M^{(G_x)},$$

i.e. $G \cdot \mu^{-1}(\beta)$ is a submanifold, and finally β is a regular value of

$$\mu|_{\mu^{-1}(G \cdot \beta)} : G \cdot \mu^{-1}(\beta) \longrightarrow G \cdot \beta.$$

Therefore $\mu^{-1}(\beta)$ is a submanifold of M and for every $z \in \mu^{-1}(\beta)$ the tangent space of $G \cdot \mu^{-1}(\beta)$ splits as

$$(5) \quad T_z G \cdot z \oplus^{\perp_\omega} T_z \mu^{-1}(\beta) = T_z G \cdot \mu^{-1}(\beta).$$

Since $T_z \mu^{-1}(\beta) = (V)^{G_x}$ and $G \cdot z$ is symplectic, we conclude that both $G \cdot \mu^{-1}(\beta)$ and $\mu^{-1}(\beta)$ are symplectic submanifolds of M . Moreover, from (1) and (5) we obtain that ϕ is a G -equivariant symplectomorphism.

□

Proof of Theorem 1.1. $((1) \iff (2))$. $(1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (1)$. Assume that the square of the moment map is constant. Let $x \in M$. By the argument used in the proof of Proposition 1.2, we have $G \cdot x$ is symplectic and $G_x = G_{\mu(x)}$. Therefore all G -orbits are symplectic $((2) \Rightarrow (5))$ and the center acts trivially on M , i.e. G is semisimple. Indeed, coadjoint orbits are of the form $G/C(T)$, where $C(T)$ is the centralizer of the torus T . In particular $Z(G) \subset G_x$ for every $x \in M$.

We want to show that the manifold M is mapped by the moment map to a single coadjoint orbit $((2) \Rightarrow (3))$.

Let $G \cdot x$ be a principal orbit. Since G_x acts trivially on the slice, from the local normal form for the moment map, in a G -invariant neighborhood of $G \cdot x$ the moment map is given by

$$\mu([g, v]) = Ad(g)(\beta).$$

This proves that there exists a G -invariant neighborhood of $G \cdot x$ which is mapped to a single coadjoint orbit. It is well-known that the set $M^{(G_x)}$ is an open dense subset of M and $M^{(G_x)}/G$ is connected ([15]). Since μ is G -equivariant, it induces a continuous application

$$\bar{\mu} : M^{(G_x)}/G \longrightarrow \mathfrak{g}/G,$$

which is locally constant. Hence $\bar{\mu}(M^{(G_x)}/G)$ is constant so $\bar{\mu}(M/G)$ is. Thus M is mapped by μ to a single coadjoint orbit; in particular $M = G \cdot \mu^{-1}(\beta)$. Note that this argument proves $(4) \Rightarrow (3)$.

Let $x \in M$. As in the proof of Proposition 1.2, from (1), (3), (4) and (5), the following application

$$\phi : G \cdot x \times \mu^{-1}(\mu(x)) \longrightarrow M, \quad (gx, z) \longrightarrow gz,$$

is the desired G -equivariant symplectomorphism.

$(2) \Rightarrow (3)$, $(2) \Rightarrow (5)$ and $(4) \Rightarrow (3)$ follow from the above discussion while $(3) \Rightarrow (2)$ and $(5) \Rightarrow (4)$ are easy to check.

$(3) \Rightarrow (5)$. In the sequel we follow the notation introduced in the proof of the Proposition 1.2. Let $G \cdot x$ be a G -orbit and let Y' be the neighborhood of the zero section in $(Y = G \times_{G_x} (\mathfrak{q} \oplus V, \tau))$ which is G -equivariant symplectomorphic to a neighborhood of $G \cdot x$. The moment map μ in Y' is given by the formula

$$\mu([g, m, v]) = Ad(g)(\beta + m + \mu_V(v)).$$

From Proposition 13 in [1], shrinking Y' if necessary, we have

$$\mu^{-1}(G \cdot \beta) \cap Y' = \{[g, m, v] : m = 0 \text{ and } \mu_V(v) = 0\}.$$

Since M is mapped by the moment map μ to a single coadjoint orbit $G \cdot \beta$, we conclude that $\mathfrak{q} = \{0\}$ and therefore $G \cdot x$ is symplectic.

Now assume that M is a Kähler manifold. Then $\omega = g(J, \cdot)$ where J is the integrable complex structure.

We shall prove that if $x \in M$, then the application

$$\phi : Gx \times \mu^{-1}(\mu(x)), \quad \phi(gx, z) = gz,$$

is an isometry.

From the argument used in the proof of Proposition 1.2, we have that for every $y \in \mu^{-1}(\mu(x))$, $T_y \mu^{-1}(\mu(x)) = T_y M^{G_x}$. Indeed, $G_y = G_x = G_{\mu(x)}$ and $G_{\mu(x)}$ centralizes a torus;

therefore $N(G_y)/G_y$ is finite. Since the complex totally geodesic submanifold M^{G_x} is given by

$$M^{G_x} = N(G_x)/G_x \times V^{G_x} = N(G_x)/G_x \times \mu^{-1}(\mu(x)),$$

where the last equality follows from the fact that ϕ is a G -equivariant symplecomorphism, we conclude that the connected component of $\mu^{-1}(\mu(x))$ which contains x is the connected component of M^{G_x} which contains x . Therefore $\mu^{-1}(\mu(x))$ is a complex totally geodesic submanifold of M .

Now, we show that all G -orbits are complex.

Let $V \in T_y \mu^{-1}(\mu(x))$ and let $X^\#$ be a tangent vector to $T_y G \cdot y$ induced by $X \in \mathfrak{g}$. Then

$$(6) \quad 0 = d\mu_y(V)(X) = \omega(X^\#, V) = g(J(X^\#), V) = g(X^\#, J(V))$$

which implies that $T_y G \cdot y = T_y \mu^{-1}(\mu(x))^\perp$ so $G \cdot y$ is complex.

Finally we prove that ϕ is an isometry. Since ϕ is G -equivariant and G acts by isometries, it is enough to prove that $d\phi_{(x,z)}$ is an isometry for every $z \in \mu^{-1}(\mu(x))$. Note that the tangent space of M splits as

$$(7) \quad T_z M = T_z G \cdot z \oplus^\perp T_z \mu^{-1}(\mu(x)),$$

for every $z \in \mu^{-1}(\mu(x))$. Hence it is sufficient to prove that the Killing vector field from $\xi \in \mathfrak{g}$ has constant norm along $\mu^{-1}(\mu(x))$.

Let $\xi \in \mathfrak{g}$ and let X be a vector field tangent to $\mu^{-1}(\mu(x))$. Then $[\xi^\#, X] = 0$ since ϕ is a G -equivariant diffeomorphism. Now, given $\eta^\#$ be such that $J(\eta^\#) = \xi^\#$, by the closeness of ω we have

$$0 = d\omega(X, \eta^\#, \xi^\#) = Xg(\xi^\#, \xi^\#)$$

which implies that the Killing field from $\xi \in \mathfrak{g}$ has constant norm along $\mu^{-1}(\mu(x))$.

In [1] it was proved that there exists a G -invariant almost complex structure J adapted to ω , i.e. $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(\cdot, J\cdot) = g$ is a Riemannian metric. Since $T_y \mu^{-1}(\mu(x)) = T_y M^{G_x}$, $\mu^{-1}(\mu(x))$ is J -invariant. This allow us to conclude that the splitting (7) holds, hence every G -orbit is J -invariant. Now, the fact ϕ is an isometry follows as before, proving the result in the almost-Kähler setting. \square

Proof of Theorem 1.3. (1) \iff (2) \iff (3) follow using the same arguments in the proof of the Theorem 1.1 while (4) \implies (3) is easy to check. We shall prove (3) \implies (4).

Let $x \in M$ be a regular point and let $\beta = \mu(x)$. As in the proof of the Theorem 1.1, $M = G \cdot \mu^{-1}(\beta)$ and any orbit is symplectic. This implies that β is a regular value of the application

$$\mu : M \longrightarrow G \cdot \beta.$$

Therefore $\mu^{-1}(\beta)$ is a closed submanifold whose tangent space is given by

$$T_y \mu^{-1}(\beta) = \text{Ker } d\mu_y = (T_y G \cdot y)^\perp$$

and the tangent space of M splits as

$$(8) \quad T_y M = T_y G \cdot y \oplus^\perp T_y \mu^{-1}(\beta).$$

for every $y \in \mu^{-1}(\beta)$. In particular $\mu^{-1}(\beta)$ is symplectic.

Now, we show that G_x acts trivially on $\mu^{-1}(\beta)$.

Note that $(G_y)^o = (G_x)^o = (G_\beta)^o$, for every $y \in \mu^{-1}(\beta)$, due the fact that $G \cdot y$ is symplectic, $G_y \subset G_\beta$ since μ is G -equivariant, and $\mu^{-1}(\beta)$ is connected since both G and $M = G \cdot \mu^{-1}(\beta)$ are.

From the slice theorem, see [15], there exists a neighborhood U of the regular point x such that $(G_z) = (G_x) \forall z \in U$. Assume that we may find a sequence $x_n \rightarrow x$ in $\mu^{-1}(\beta)$ and a sequence $g_n \in G_{x_n} - G_x \subseteq G_\beta$ such that $g_n x_n = x_n$. Since the G -action is proper we may assume that $g_n \rightarrow g_o$ which lies in G_x . In particular the sequence g_n converges to g_o in G_β as well, since it is a closed Lie group.

Now, G_x is an open subset of G_β , since $(G_x)^o = (G_\beta)^o$; therefore there exists n_o such that $g_n \in G_x$ for $n \geq n_o$ which is an absurd.

Thus, there exists an open subset U' of x in $\mu^{-1}(\beta)$ such that $G_z = G_x, \forall z \in U'$. Since G_x is compact we conclude that G_x acts trivially on $\mu^{-1}(\beta)$.

It follows that the application

$$\phi : G \cdot x \times \mu^{-1}(\beta) \longrightarrow M, \quad \phi(gG_x, z) = gz$$

is well-defined, smooth and G -equivariant. Moreover, from (1) and (8) we get that

$$(9) \quad \phi^*(\omega) = \omega|_{G \cdot x} + \omega|_{\mu^{-1}(\beta)}.$$

Note also $Z(G)^o \subseteq (G_{\mu(x)})^o = (G_x)^o$, so G must be semisimple whenever G is a reductive Lie group acting effectively on M and ϕ becomes a symplectomorphism from Proposition 2.1 since a G -orbit intersects $\mu^{-1}(\beta)$ in one point.

Now assume M is almost-Kähler and $N(G_x)/G_x$ is finite. The same arguments used in the proof of Theorem 1.1 show that $\mu^{-1}(\beta) \cap M^{G_x}$ is complex, $G \cdot y$ is complex for every $y \in \mu^{-1}(\beta) \cap M^{G_x}$ and the following map

$$\bar{\phi} = \phi|_{G \cdot x \times (\mu^{-1}(\beta) \cap M^{G_x})} : G \cdot x \times (\mu^{-1}(\beta) \cap M^{G_x}) \longrightarrow M^{(G_x)}, \quad \phi([gx, z]) = gz,$$

is G -equivariant and it satisfies (9); therefore a local diffeomorphism. Then we may use the same argument in the proof of the Theorem 1.1 to prove that $\bar{\phi}$ is an isometry. Since $M^{(G_x)}$ is an open dense subset of M , we obtain that $\mu^{-1}(\beta)$ is complex, so all G -orbits are, since the symplectic splitting (7) turns out to be g -orthogonal, where $g = \omega(\cdot, J\cdot)$ is the induced Riemannian metric, and finally we get ϕ is an isometry. □

Proof of Corollary 1.4 We recall that an element $X \in \mathfrak{g}$ is called *elliptic* if $\text{ad}(X) \in \text{End}(\mathfrak{g}^{\mathbb{C}})$ is diagonalizable and all eigenvalues are purely imaginary. Then $G \cdot X$ is called an *elliptic orbit*. See [13] and [12] for more details about elliptic orbits.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra of G . Since any elliptic element is conjugate to an element of \mathfrak{k} , we have that the squared moment map $f = \|\mu\|^2$ is positive. Therefore, using the same argument in the proof of Theorem 1.2, we conclude that all G -orbits are symplectic. Now the result follows from Theorem 1.3. □

REFERENCES

- [1] L. BATES AND E. LERMAN, Proper group actions and symplectic stratified spaces, Pacific J. Math. 181 (1997) 201-229.
- [2] L. BEDULLI AND A. GORI, A splitting result for compact symplectic manifolds, Results Math 47(3) (2005) 194-198.
- [3] A. L. BESSE, Einstein manifold, Springer Verlag, New York, 1978.

- [4] A. S. CANNAS, Lectures on Symplectic Geometry, Lecture Notes in Math. 1764, Springer-Verlag, Berlin, 2001.
- [5] J. CHEEGER AND D. EBIN, Comparison theorems in Riemannian geometry, North-Holland Publishing Co., Amsterdam, 1975.
- [6] A. GORI AND F. PODESTÀ, A note on the moment map on compact Kähler manifold, Ann. Global. Anal. Geom. 26(3) (2004) 315–318.
- [7] V. GUILLEMIN AND S. STERNBERG, Symplectic techniques in physics, Cambridge University Press, 1984.
- [8] S. HELGASON, Differential Geometry, Lie groups, and symmetric spaces, Academic Press, New-York and London, second edition, 1978.
- [9] A. HUCKLEBERRY AND T. WURZBACHER, Multiplicity-free complex manifolds, Math. Ann. 286 (1990) 261–280.
- [10] F. KIRWAN, Cohomology quotients in symplectic and algebraic geometry, Math. Notes 31, Princeton, 1984.
- [11] S. KOBAYASHI AND N. NOMIZU, Foundations of Differential Geometry, Vol. I, Interscience Publisher, J. Wiles & Sons, 1963.
- [12] T. KOBAYASHI, Lectures on restrictions of unitary representations of real reductive groups, RIMS Preprint **1526**, University of Kyoto, 2005.
- [13] K.-H. NEEB, Holomorphy and Convexity in Lie theory, Expositions in Mathematics, de Gruyter, Berlin, 1999.
- [14] J. P. ORTEGA AND T. S. RATIU, A symplectic slice Theorem, Lett. Math. Phys. 58 (2002) 81–93.
- [15] R. S. PALAIS AND C. L. TERNG, Critical point theory and submanifold geometry, Lecture Notes in Math. 1353, Springer-Verlag, Berlin, 1988.
- [16] R. SJAMAAR AND E. LERMAN, Stratified symplectic spaces and reduction, Ann. of Math. 134 (1991) 375–422.
- [17] L. PUKANSZKY, Unitary representations of solvable groups, Ann. Sci. École Norm. Sup. **4** (1972) 475–608.

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